

RATIONAL SURFACES IN INDEX-ONE FANO HYPERSURFACES

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ABSTRACT. We give the first evidence for a conjecture that a general, index-one, Fano hypersurface is not unirational: (i) a general point of the hypersurface is contained in no rational surface ruled, roughly, by low-degree rational curves, and (ii) a general point is contained in no image of a Del Pezzo surface.

1. INTRODUCTION

For complex, projective varieties a classical notion is unirationality: A variety rationally dominated by projective space is *unirational*. A modern notion is rational connectedness: A variety is *rationally connected* if every pair of points is contained in a rational curve. Every unirational variety is rationally connected. The 2 notions agree for curves and surfaces. Conjecturally they disagree in higher dimensions.

Conjecture 1.1. For every integer $n \geq 4$ there exists a non-unirational, smooth, degree- n hypersurface in \mathbb{P}^n .

A smooth hypersurface in \mathbb{P}^n of degree $d \leq n$ is an *index- $(n+1-d)$, Fano manifold*. By [2] and [9], every Fano manifold is rationally connected. Versions of Conjecture 1.1 have been around for decades. The specific case $n = 4$ is attributed to Iskovskikh and Manin, [7].

In [8], Kollár suggested an approach to proving Conjecture 1.1. A general point of an n -dimensional, unirational variety is contained in a k -dimensional, rational subvariety for each $k < n$. Thus, Conjecture 1.1 for $n \geq 5$ follows from the next conjecture (which fails for $n = 4$).

Conjecture 1.2. For every integer $n \geq 5$, there exists a smooth, degree- n hypersurface in \mathbb{P}^n whose general point is contained in no rational surface.

We give the first evidence for Conjecture 1.2.

Theorem 1.3. *For every integer $n \geq 5$, every smooth, degree- n hypersurface X in \mathbb{P}^n contains a countable union of closed, codimension-2 subvarieties containing the image of every generically-finite, rational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ mapping a general fiber $\{t\} \times \mathbb{P}^1$ isomorphically to an $(n-1)$ -normal, smooth, rational curve in X .*

Theorem 1.4. *For every integer $n \geq 5$, every smooth, degree- n hypersurface in \mathbb{P}^n contains a countable union of closed, codimension-2 subvarieties containing the image of every generically finite, regular morphism from a Del Pezzo surface to X .*

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A projective variety is *k-normal* if every global section of the restriction of $\mathcal{O}_{\mathbb{P}^n}(k)$ is the restriction of a global section on \mathbb{P}^n .

There are 2 approaches. First, given a rational surface S and a regular morphism $f : S \rightarrow X$, to prove deformations of f are contained in a codimension-2 subvariety of X , it suffices to prove $\bigwedge^{n-4}(f^*T_X/T_S)/\text{Torsion}$ has only the zero section. In Section 2, this is used to prove Theorems 1.3 and 1.4.

Second, a rational surface with a pencil of rational curves gives a morphism from \mathbb{P}^1 to a parameter space of rational curves on X . There is a canonical construction of algebraic differential forms on the parameter space. Since \mathbb{P}^1 has only the zero form, these forms limit rational curves on the parameter space. In Section 3, this is used to prove the following generalization of Theorem 1.3.

Theorem 1.5. *For every integer $n \geq 5$, every smooth, degree- n hypersurface X in \mathbb{P}^n contains a countable union of closed, codimension-2 subvarieties containing the image of every generically-finite, rational transformation $S \dashrightarrow X$ from a surface with a pencil of curves mapping the general curve isomorphically to an $(n-1)$ -normal, smooth curve of genus 0 or 1, also assumed non-degenerate if the genus is 1.*

As the second approach does not apply to Theorem 1.4, the first approach is more productive. However, further progress in proving Conjecture 1.2 will likely use both approaches, as well as new ideas.

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2. THE FIRST APPROACH

Let X be a smooth, degree- n hypersurface in \mathbb{P}^n , $n \geq 5$. Denote by $\text{Hilb}(X)$ the Hilbert scheme of X . Theorem 1.3 follows easily from the next theorem.

Theorem 2.1. *Let Z be an irreducible subscheme of $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$ satisfying,*

- (i) *the associated morphism $Z \times \mathbb{P}^1 \rightarrow \text{Hilb}(X)$ does not factor through the projection $Z \times \mathbb{P}^1 \rightarrow Z$, and*
- (ii) *the image of a general point of $Z \times \mathbb{P}^1$ parametrizes a smooth, $(n-1)$ -normal, rational curve in X .*

Then there exists a codimension ≥ 2 subvariety of X containing all curves parametrized by $Z \times \mathbb{P}^1$.

A morphism $\mathbb{P}^1 \rightarrow \text{Hilb}(X)$ is equivalent to a closed subscheme $S' \subset \mathbb{P}^1 \times X$, flat over \mathbb{P}^1 . If a general point of \mathbb{P}^1 parametrizes a smooth rational curve, then S' is an irreducible surface. Any desingularization S of S' is a surface fitting in a diagram,

$$\begin{array}{ccc} S & \xrightarrow{f} & X \\ \downarrow \pi & & \\ \mathbb{P}^1 & & \end{array} \tag{1}$$

Proposition 2.2. *Let Z be an irreducible subvariety of $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$ satisfying,*

- (i) *the associated morphism $Z \times \mathbb{P}^1 \rightarrow \text{Hilb}(X)$ does not factor through the projection $Z \times \mathbb{P}^1 \rightarrow Z$,*
- (ii) *a general point of $Z \times \mathbb{P}^1$ parametrizes a smooth curve in X , and*
- (iii) *there is no codimension 2 subvariety of X containing all curves parametrized by $Z \times \mathbb{P}^1$.*

*Then, for the morphism $\mathbb{P}^1 \rightarrow \text{Hilb}(X)$ parametrized by a general point of Z , the torsion-free sheaf $\bigwedge^{n-4}(f^*T_X/T_S)/\text{Torsion}$ has a nonzero global section.*

Proof. Replacing Z by a dense, Zariski open subset if necessary, assume Z is smooth. Let $V \subset Z \times \mathbb{P}^1 \times X$ be the pullback of the universal family to $Z \times \mathbb{P}^1$

Let $g : \tilde{V} \rightarrow V$ be a desingularization of V . Denote by ϕ the projection map from V to $Z \times X$, and denote by p the projection map from V to $Z \times \mathbb{P}^1$. Let $\tilde{\phi} = \phi \circ g$, and let $\tilde{p} = p \circ g$.

$$\begin{array}{ccccc}
 & & \tilde{g} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \tilde{V} & \xrightarrow{g} & V & \xrightarrow{\phi} & Z \times X \\
 & \searrow & \downarrow & \nearrow & \\
 & & Z \times \mathbb{P}^1 & & \\
 & \searrow \tilde{p} & \downarrow p & \nearrow & \\
 & & Z & &
 \end{array}$$

Replacing Z by a dense, Zariski open subset if necessary, assume \tilde{p} is smooth, cf. [6, Corollary III.10.7]. Associated to the morphism \tilde{g} is the derivative map,

$$d\tilde{g} : T_{\tilde{V}} \rightarrow \tilde{g}^*T_X.$$

Associated to the morphism \tilde{p} is the derivative map,

$$d\tilde{p} : T_{\tilde{V}} \rightarrow \tilde{p}^*T_Z.$$

By hypothesis, $d\tilde{p}$ is surjective. Denote by $T_{\tilde{p}}$ the kernel of $d\tilde{p}$. Because $Z \times \mathbb{P}^1 \rightarrow \text{Hilb}(X)$ does not factor through Z , the restriction of \tilde{g} to a general fiber of \tilde{p} maps generically finitely to its image. Therefore the following map is generically injective,

$$d\tilde{g} : T_{\tilde{p}} \rightarrow \tilde{g}^*T_X.$$

As $T_{\tilde{p}}$ is locally free and \tilde{V} is integral, in fact $d\tilde{g}$ is injective. Denoting by \mathcal{N} the cokernel of \tilde{g}^*T_X by $d\tilde{g}(T_{\tilde{p}})$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_{\tilde{p}} & \longrightarrow & T_{\tilde{V}} & \xrightarrow{d\tilde{p}} & \tilde{p}^*T_Z \longrightarrow 0 \\
 & & \downarrow = & & \downarrow d\tilde{g} & & \downarrow u \\
 0 & \longrightarrow & T_{\tilde{p}} & \longrightarrow & \tilde{g}^*T_X & \longrightarrow & \mathcal{N} \longrightarrow 0
 \end{array}$$

By generic smoothness, the rank of $d\tilde{g}$ at a general point of \tilde{V} equals the dimension of the closure of $\text{Image}(\tilde{g})$. By hypothesis, this is $\geq n - 2$. Therefore the rank of u at a general point is $\geq n - 4$. Thus, the restriction of u to a general $(n - 4)$ -plane in the fiber of \tilde{p}^*T_Z has rank $n - 4$. A general $(n - 4)$ -plane is the tangent space

of a general $(n - 4)$ -dimensional subvariety of Z . Therefore, after replacing Z by the smooth locus of a general $(n - 4)$ -dimensional subvariety of Z , assume Z is $(n - 4)$ -dimensional and u is generically injective.

Associated to u , there is an induced map,

$$\bigwedge^{n-4}(u) : \tilde{p}^* \bigwedge^{n-4} T_Z \rightarrow \bigwedge^{n-4} \mathcal{N}/\text{Torsion}.$$

Because u is generically injective and $n \geq 5$, this map is generically injective.

Let z be a general point of Z , and denote by S and \tilde{S} the fibers of p and \tilde{p} over z , respectively. Since \tilde{V} is smooth, \tilde{S} is a smooth surface. Let $f : S \rightarrow X$ be the restriction of ϕ to S , and let $\tilde{f} : \tilde{S} \rightarrow X$ be the corresponding map from \tilde{S} . The restriction of \mathcal{N} to S is precisely f^*T_X/T_S . The restriction of \tilde{p}^*T_Z to S is precisely the trivial vector bundle $T_{Z,z} \otimes_{\mathbb{C}} \mathcal{O}_S$. Since z is general, the restriction of $\bigwedge^{n-4}(u)$ is generically injective. Therefore, it is a nonzero map,

$$\bigwedge^{n-4}(u)|_S : \left(\bigwedge^{n-4} T_{Z,z} \right) \otimes_{\mathbb{C}} \mathcal{O}_S \rightarrow \left(\bigwedge^{n-4} f^*T_X/T_S \right) / \text{Torsion}.$$

Since $T_{Z,z}$ is $(n - 4)$ -dimensional, this is equivalent to a nonzero global section of $(\bigwedge^{n-4} f^*T_X/T_S)/\text{Torsion}$ (well-defined up to nonzero scaling). \square

Proposition 2.3. *Let $\mathbb{P}^1 \rightarrow \text{Hilb}(X)$ be a morphism with associated diagram as in Equation 1. If the curve parametrized by a general point of \mathbb{P}^1 is smooth and $(n - 1)$ -normal then*

$$h^0(S, \omega_S \otimes \bigwedge^{n-2} f^*T_X) = 0.$$

Proof. Pulling back the short exact sequence of tangent bundles

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^n}|_X \rightarrow N_{X/\mathbb{P}^n} \cong \mathcal{O}_X(n) \rightarrow 0$$

to S and taking its $(n - 1)^{\text{st}}$ exterior power gives another short exact sequence

$$0 \rightarrow \bigwedge^{n-1} f^*T_X \rightarrow \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \rightarrow f^*\mathcal{O}_X(n) \otimes \bigwedge^{n-2} f^*T_X \rightarrow 0.$$

Tensoring this sequence with $\omega_S \otimes f^*\mathcal{O}_X(-n)$ gives the following short exact sequence

$$0 \rightarrow \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X \rightarrow \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \rightarrow \omega_S \otimes \bigwedge^{n-2} f^*T_X \rightarrow 0.$$

Applying the long exact sequence of cohomology, $h^0(S, \omega_S \otimes \bigwedge^{n-2} f^*T_X)$ equals 0 if both,

- (i) $h^0(S, \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n})$ equals 0, and
- (ii) $h^1(S, \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X)$ equals 0.

Proof of (i). Consider the Euler exact sequence on \mathbb{P}^n

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

Pulling this back to S , and taking its n^{th} exterior power gives the following exact sequence

$$0 \rightarrow \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \rightarrow f^*\mathcal{O}_X(n)^{\oplus(n+1)} \rightarrow f^*\mathcal{O}_X(n+1) \rightarrow 0.$$

Tensoring with $\omega_S \otimes f^*\mathcal{O}_X(-n)$ gives an injective map

$$\omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \rightarrow \omega_S^{\oplus(n+1)}.$$

Thus it suffices to prove $h^0(S, \omega_S)$ equals 0, which follows from the hypothesis that S is a rational surface.

Proof of (ii). There is a canonical isomorphism

$$\omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X \cong \omega_S \otimes f^*\mathcal{O}_X(-n+1).$$

So by Serre duality, it suffices to prove $h^1(S, f^*\mathcal{O}_X(n-1))$ equals 0. Let C be a general fiber of the map $\pi : S \rightarrow \mathbb{P}^1$. There is a short exact sequence

$$0 \rightarrow f^*\mathcal{O}_X(n-1) \otimes I_{C/S} \rightarrow f^*\mathcal{O}_X(n-1) \rightarrow f^*\mathcal{O}_X(n-1)|_C \rightarrow 0, \quad (2)$$

where $I_{C/S}$ is the ideal sheaf of C in S . By hypothesis, the image of C by f is $(n-1)$ -normal in \mathbb{P}^n , therefore the map

$$H^0(S, f^*\mathcal{O}_X(n-1)) \rightarrow H^0(C, f^*\mathcal{O}_X(n-1)|_C)$$

is surjective. The long exact sequence of cohomology to the sequence in Equation 2 gives an isomorphism

$$H^1(S, f^*\mathcal{O}_X(n-1) \otimes I_{C/S}) \cong H^1(S, f^*\mathcal{O}_X(n-1)). \quad (3)$$

Because S is a smooth surface and the general fiber of π is a smooth, rational curve, $R^1\pi_*\mathcal{F}$ is the zero sheaf for every π -relatively globally-generated, coherent \mathcal{O}_S -module \mathcal{F} . Because $f^*\mathcal{O}_X(n-1)$ is globally-generated, it is π -relatively globally-generated. Because $I_{C/S}$ equals $\pi^*\mathcal{O}_{\mathbb{P}^1}(-1)$, the twist $f^*\mathcal{O}_X(n-1) \otimes I_{C/S}$ is π -relatively globally-generated. Thus $R^1\pi_*(f^*\mathcal{O}_X(n-1))$ and $R^1\pi_*(f^*\mathcal{O}_X(n-1) \otimes I_{C/S})$ are each the zero sheaf. So, by the Leray spectral sequence, there are canonical isomorphisms

$$H^1(S, f^*\mathcal{O}_X(n-1)) \cong H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1))), \quad (4)$$

$$H^1(S, f^*\mathcal{O}_X(n-1) \otimes I_{C/S}) \cong H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1) \otimes I_{C/S})). \quad (5)$$

Taken together, Equations 3, 4 and 5 give a canonical isomorphism

$$H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1))) \cong H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1))).$$

This is possible only if $h^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1)))$ equals 0. \square

Proof of Theorem 2.1. Let Z satisfy the hypotheses of Proposition 2.2, and let S and f satisfy the conclusion of Proposition 2.2. The injective map $df : T_S \rightarrow f^*T_X$ induces a multiplication map,

$$\bigwedge^2 T_S \otimes \bigwedge^{n-4} f^*T_X \rightarrow \bigwedge^{n-2} f^*T_X.$$

The image sheaf is precisely $\bigwedge^2 T_X \otimes (\bigwedge^{n-4}(f^*T_X/T_S))/\text{Torsion}$. Tensoring with the canonical bundle of ω_S , this gives an injective map

$$(\bigwedge^{n-4}(f^*T_X/T_S))/\text{Torsion} \rightarrow \omega_S \otimes \bigwedge^{n-2} f^*T_X.$$

By hypothesis, $(\bigwedge^{n-4}(f^*T_X/T_S))/\text{Torsion}$ has a nonzero global section. Therefore $\omega_S \otimes \bigwedge^{n-2} f^*T_X$ also has a nonzero global section.

On the other hand, for Z satisfying the hypothesis of Theorem 2.1, Proposition 2.3 implies,

$$h^0(S, \omega_S \otimes \bigwedge^{n-2} f^* T_X) = 0.$$

Thus Z does not satisfy the hypothesis of Proposition 2.3, i.e., it does not satisfy Hypothesis (iii). Therefore there exists a codimension 2 subvariety of X containing all the curves parametrized by $Z \times \mathbb{P}^1$. \square

Proof of Theorem 1.3. For every generically-finite, rational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ restricting to a closed immersion on a general fiber, there is an associated rational transformation

$$\mathbb{P}^1 \dashrightarrow \text{Hilb}(X), \quad t \mapsto \text{Image}(\{t\} \times \mathbb{P}^1).$$

By properness of the Hilbert scheme and the valuative criterion, this extends to a regular morphism. Therefore, associated to each rational transformation is an element in the Hom-scheme $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$. Those rational transformations satisfying the hypothesis of Theorem 1.3 give a locally closed subset of $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$. As $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$ is a countable union of quasi-projective varieties, this subset is also a countable union of quasi-projective subvarieties. By Theorem 2.1, for each such subvariety Z , there is a codimension-2 subvariety of X containing every curve parametrized by $Z \times \mathbb{P}^1$. This subvariety contains the image of each rational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ giving a point in Z . Therefore, there exists a countable union of codimension-2 subvarieties of X containing the image of every rational transformation satisfying the hypothesis of Theorem 2.1. \square

The proof of the Theorem 1.4 is similar to the proof of Theorem 2.1. There is a preliminary proposition.

Proposition 2.4. *Let X be a smooth hypersurface of degree n in \mathbb{P}^n . For every Del Pezzo surface S and every generically finite morphism $f : S \rightarrow X$, the only global section of $\bigwedge^{n-4}(f^* T_X / T_S) / \text{Torsion}$ is the zero section.*

Proof. The proof is similar to the proof of Theorem 2.1. By the same type of argument, it suffices to prove,

- (i) $h^0(S, \omega_S \otimes f^* \mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^* T_{\mathbb{P}^n})$ equals 0, and
- (ii) $h^1(S, \omega_S \otimes f^* \mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^* T_X)$ equals 0.

The proof of (i) is the same as in the proof of Theorem 2.1, since $h^0(S, \omega_S)$ equals 0. As for (ii), there is a canonical isomorphism

$$\omega_S \otimes f^* \mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^* T_X \cong (\omega_S^{-1} \otimes f^* \mathcal{O}_X(n-1))^{-1}.$$

Denote $\omega_S^{-1} \otimes f^* \mathcal{O}_X(n-1)$ by L . The sheaf $f^* \mathcal{O}_X(n-1)$ is globally generated. By the hypothesis that S is a Del Pezzo surface, ω_S^{-1} is ample. Thus L is ample. By Kodaira vanishing, $h^1(S, L^{-1})$ equals 0. So, using the canonical isomorphism, $h^1(S, \omega_S \otimes f^* \mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^* T_X)$ equals 0. \square

Proof of Theorem 1.4. By the same countability argument as at the beginning of the section, it suffices to prove that for every flat family

$$\begin{array}{ccc} D & \xrightarrow{\phi} & X \times V \\ \downarrow p & & \\ V & & \end{array}$$

such that ϕ is generically finite and a general fiber of p is a Del Pezzo surface, the image of ϕ is contained in a subvariety of codimension ≥ 2 . Let S be the fiber of p over a general point of V , and let f be the restriction $\phi|_S : S \rightarrow X$. As in the proof of the Proposition 2.2, if the image of ϕ is contained in no subvariety of codimension ≥ 2 , then $H^0(S, \bigwedge^{n-4}(f^*T_X/T_S)/\text{Torsion})$ has a nonzero global section. Thus Proposition 2.4 proves the image of ϕ is contained in a subvariety of codimension ≥ 2 . \square

3. THE SECOND APPROACH

Let X be a smooth hypersurface in \mathbb{P}^n of degree n . We denote by $\overline{\mathcal{M}}_{g,n}(X)$ the Kontsevich moduli stack of families of genus- g , n -pointed, stable maps to X . The associated coarse moduli space is denoted by $\overline{M}_{g,n}(X)$, cf. [4].

For every integral subscheme M of $\overline{M}_{g,0}(X)$, denote by $X(M)$ the smallest closed subvariety of X containing every curve parametrized by M .

Hypothesis 3.1. Let M be an integral, closed subscheme of $\overline{M}_{g,0}(X)$.

- (i) The curves parametrized by M are contained in no codimension 2 subscheme of X , i.e., $\dim(X(M)) \geq n - 2$.
- (ii) The integer g equals 0 or 1.
- (iii) A general point of M parametrizes an embedded, smooth, $(n - 1)$ -normal curve. If g equals 1, also the curve is nondegenerate.
- (iv) The dimension of M equals $\dim(X) - 2 = n - 3$.

Theorem 3.2. *For every integer $n \geq 4$, for every smooth, degree- n hypersurface X in \mathbb{P}^n , and for every integral closed subscheme M of $\overline{M}_{g,0}(X)$ satisfying Hypothesis 3.1, every desingularization of M has a nonzero canonical form. In particular, M is not uniruled.*

There are 2 components of the proof: a global construction of certain $(n - 3)$ -forms on M following [3], and a local description of the forms proving they are nonzero. The construction is 3.3, the local description is Lemma 3.4, and the nonvanishing result is Claim 3.5.

Let \widetilde{M} be a finite type scheme and let $\nu : \widetilde{M} \rightarrow \overline{M}_{g,0}(X)$ be a morphism. Later, \widetilde{M} will be a desingularization of an integral subscheme M of $\overline{M}_{g,0}(X)$. For every integer p , [3, Corollary 4.3] gives a map,

$$\psi_p : H^1(X, \Omega_X^{p+1}) \rightarrow H^0(\overline{\mathcal{M}}_{g,0}(X), \Omega_{\overline{\mathcal{M}}_{g,0}(X)}^p).$$

Denote,

$$\widetilde{M}_{\text{stack}} := \left(\widetilde{M} \times_{\overline{M}_{g,0}(X)} \overline{\mathcal{M}}_{g,0}(X) \right)_{\text{red}},$$

i.e., the associated reduced stack of the 2-fibered product. There are projections,

$$\begin{aligned}\pi_1 : \widetilde{M}_{\text{stack}} &\rightarrow \widetilde{M}, \\ \pi_2 : \widetilde{M}_{\text{stack}} &\rightarrow \overline{\mathcal{M}}_{g,0}(X).\end{aligned}$$

There are associated pullback maps on p -forms,

$$\begin{aligned}\pi_1^* : H^0(\widetilde{M}, \Omega_{\widetilde{M}}^p) &\rightarrow H^0(\widetilde{M}_{\text{stack}}, \Omega_{\widetilde{M}_{\text{stack}}}^p), \\ \pi_2^* : H^0(\overline{\mathcal{M}}_{g,0}(X), \Omega_{\overline{\mathcal{M}}_{g,0}(X)}^p) &\rightarrow H^0(\widetilde{M}_{\text{stack}}, \Omega_{\widetilde{M}_{\text{stack}}}^p).\end{aligned}$$

Hence there is a map,

$$\pi_2^* \circ \psi : H^1(X, \Omega_X^{p+1}) \rightarrow H^0(\widetilde{M}_{\text{stack}}, \Omega_{\widetilde{M}_{\text{stack}}}^p).$$

Does this map factor through π_1^* , i.e., does there exist a map,

$$\phi_p : H^1(X, \Omega_X^{p+1}) \rightarrow H^0(\widetilde{M}, \Omega_{\widetilde{M}}^p),$$

such that $\pi_1^* \circ \phi_p$ equals $\pi_2^* \circ \psi_p$?

Lemma 3.3. *If \widetilde{M} is smooth there is a unique map of \mathbb{C} -vector spaces,*

$$\phi_p : H^1(X, \Omega_X^{p+1}) \rightarrow H^0(\widetilde{M}, \Omega_{\widetilde{M}}^p),$$

such that $\pi_1^ \circ \phi_p$ equals $\pi_2^* \circ \psi_p$.*

Proof. Since \widetilde{M} is smooth and $\widetilde{M}_{\text{stack}}$ is reduced, [3, Proposition 3.6] implies the pullback map,

$$H^0(\widetilde{M}, \Omega_{\widetilde{M}}^p) \rightarrow H^0(\widetilde{M}_{\text{stack}}, \Omega_{\widetilde{M}_{\text{stack}}}^p),$$

is an isomorphism. \square

Let C be a smooth curve in X with corresponding point $[C]$ in $\overline{\mathcal{M}}_{g,0}(X)$. For every integer p , restriction to the fiber at $[C]$ defines a map,

$$\psi_{p,[C]} : H^1(X, \Omega_X^{p+1}) \rightarrow \Omega_{\overline{\mathcal{M}}_{g,0}(X)}^p|_{[C]}.$$

The Zariski tangent space to $\overline{\mathcal{M}}_{g,0}(X)$ at $[C]$ is $H^0(C, N_{C/X})$. The dual vector space is the fiber of $\Omega_{\overline{\mathcal{M}}_{g,0}(X)}^p$ at $[C]$. The p^{th} exterior power is the fiber of $\Omega_{\overline{\mathcal{M}}_{g,0}(X)}^p$ at $[C]$. Therefore $\psi_{p,[C]}$ is equivalent to a linear map,

$$\psi_{p,[C]} : H^1(X, \Omega_X^{p+1}) \rightarrow \text{Hom}\left(\bigwedge^p H^0(C, N_{C/X}), \mathbb{C}\right).$$

What is the map $\psi_{p,[C]}$? In other words, for an element in $H^1(X, \Omega_X^{p+1})$, what is the associated p -linear alternating map on $H^0(C, N_{C/X})$?

When X is a smooth hypersurface in \mathbb{P}^n and $p = n - 3$, the answer follows as in [3, Theorem 5.1]. For a smooth hypersurface X in \mathbb{P}^n , Griffiths computed the cohomology groups $H^q(X, \Omega_X^p)$, cf. [5, Section 8]. In particular, there is an exact sequence,

$$H^0(X, \Omega_{\mathbb{P}^n}^{n-1}(X)|_X) \rightarrow H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X) \rightarrow H^1(X, \Omega_X^{n-2}) \rightarrow 0. \quad (6)$$

Therefore every element of $H^1(X, \Omega_X^{n-2})$ is the image $\overline{\beta}$ of an element β in $H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X)$.

There is a short exact sequence of locally free \mathcal{O}_C -modules,

$$0 \rightarrow N_{C/X} \rightarrow N_{C/\mathbb{P}^n} \rightarrow N_{X/\mathbb{P}^n}|_C \rightarrow 0.$$

Taking the $(n-2)^{\text{nd}}$ exterior power gives a short exact sequence,

$$0 \rightarrow \bigwedge^{n-2} N_{C/X} \rightarrow \bigwedge^{n-2} N_{C/\mathbb{P}^n} \rightarrow (\bigwedge^{n-3} N_{C/X}) \otimes N_{X/\mathbb{P}^n}|_C \rightarrow 0.$$

Twisting each term by $N_{X/\mathbb{P}^n}^\vee|_C$ gives an exact sequence,

$$0 \rightarrow \bigwedge^{n-2} N_{C/X} \otimes N_{X/\mathbb{P}^n}^\vee|_C \rightarrow \bigwedge^{n-2} N_{C/\mathbb{P}^n} \otimes N_{X/\mathbb{P}^n}^\vee|_C \rightarrow \bigwedge^{n-3} N_{C/X} \rightarrow 0. \quad (7)$$

Applying the long exact sequence of cohomology, there is a connecting map,

$$H^0(C, \bigwedge^{n-3} N_{C/X}) \rightarrow H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N_{X/\mathbb{P}^n}^\vee|_C).$$

Now $\bigwedge^{n-2} N_{C/X}$ is the determinant of $N_{C/X}$, which is canonically isomorphic to $\omega_C \otimes (\Omega_X^{n-1})^\vee|_C$. By adjunction, Ω_X^{n-1} is isomorphic to $\Omega_{\mathbb{P}^n}^n(X)|_X$. Also, N_{X/\mathbb{P}^n} is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(X)|_X$. Putting this together gives a canonical isomorphism,

$$\bigwedge^{n-2} N_{C/X} \otimes N_{X/\mathbb{P}^n}^\vee|_C \cong \omega_C \otimes (\Omega_{\mathbb{P}^n}^n(2X)^\vee)|_C.$$

Serre duality gives an isomorphism,

$$H^1(C, \omega_C \otimes (\Omega_{\mathbb{P}^n}^n(2X)^\vee)|_C) \cong H^0(C, \Omega_{\mathbb{P}^n}^n(2X)|_C)^\vee.$$

The pullback map $H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X) \rightarrow H^0(C, \Omega_{\mathbb{P}^n}^n(2X)|_C)$ determines a transpose map,

$$H^0(C, \Omega_{\mathbb{P}^n}^n(2X)|_C)^\vee \rightarrow H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X)^\vee.$$

Finally, every element β of $H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X)$ determines a linear functional,

$$H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X)^\vee \rightarrow \mathbb{C}.$$

Putting all this together, every element β of $H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X)$ determines a linear functional,

$$\tilde{\beta} : H^0(C, \bigwedge^{n-3} N_{C/X}) \rightarrow \mathbb{C}.$$

Lemma 3.4. *Let X be a smooth hypersurface in \mathbb{P}^n . For every element β of $H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X)$, $\phi_{n-3, [C]}(\tilde{\beta})$ equals the restriction of $\tilde{\beta}$ to $\bigwedge^{n-2} H^0(C, N_{C/X})$, up to nonzero scaling.*

Proof. This follows by a diagram-chase. Here are the main points. There is a commutative diagram with exact rows and columns,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & (8) \\
 & & \downarrow & & \downarrow & & \\
 & & T_C & \xrightarrow{\text{Id}} & T_C & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & T_X|_C & \longrightarrow & T_{\mathbb{P}^n}|_C & \longrightarrow & N_{X/\mathbb{P}^n}|_C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{Id} \\
 0 & \longrightarrow & N_{C/X} & \longrightarrow & N_{C/\mathbb{P}^n} & \longrightarrow & N_{X/\mathbb{P}^n}|_C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The following three invertible sheaves are isomorphic,

$$\omega_C \otimes \bigwedge^{n-2} N_{C/X}^\vee \cong \Omega_X^{n-1}|_C \cong \Omega_{\mathbb{P}^n}^n(X)|_C.$$

Denote any by L . Twisting Equation 8 by L gives a commutative diagram with exact rows and columns,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \bigwedge^{n-2} N_{C/X}^\vee & \xrightarrow{\cong} & \bigwedge^{n-1} N_{C/\mathbb{P}^n}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_X^{n-2}|_C & \longrightarrow & \Omega_{\mathbb{P}^n}^{n-1}(X)|_C & \longrightarrow & \Omega_X^{n-1}|_C \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong \\
 0 & \longrightarrow & \omega_C \otimes \bigwedge^{n-3} N_{C/X}^\vee & \longrightarrow & \omega_C \otimes \bigwedge^{n-2} N_{C/\mathbb{P}^n}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_C & \longrightarrow & \omega_C \otimes \bigwedge^{n-2} N_{C/X}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{9}$$

For a sheaf \mathcal{E} on X , $\mathcal{E}(X)|_X$ denotes the tensor product $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_X$. And for a sheaf \mathcal{F} on C , $\mathcal{F}(X)|_C$ denotes the tensor product, $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_C$.

Consider the last map in the first column of Equation 9. There is an associated map of cohomology groups $H^1(C, -)$,

$$H^1(C, \Omega_X^{n-2}|_C) \rightarrow H^1(C, \omega_C \otimes (\bigwedge^{n-3} N_{C/X})^\vee).$$

By Serre duality this is equivalent to,

$$H^1(C, \Omega_X^{n-2}|_C) \rightarrow \text{Hom}(H^0(C, \bigwedge^{n-3} N_{C/X}), \mathbb{C}).$$

There is a natural multiplication map,

$$\bigwedge^{n-3} H^0(C, N_{C/X}) \rightarrow H^0(C, \bigwedge^{n-3} N_{C/X}),$$

having transpose,

$$\mathrm{Hom}(H^0(C, \bigwedge^{n-3} N_{C/X}), \mathbb{C}) \rightarrow \mathrm{Hom}(\bigwedge^{n-3} H^0(C, N_{C/X}), \mathbb{C}).$$

Composing the previous map with the transpose gives a map,

$$H^1(C, \Omega_X^{n-2}|_C) \rightarrow \mathrm{Hom}(\bigwedge^{n-3} H^0(C, N_{C/X}), \mathbb{C}).$$

Composing the restriction map with the last map gives a map,

$$H^1(X, \Omega_X^{n-2}) \rightarrow \mathrm{Hom}(\bigwedge^{n-3} H^0(C, N_{C/X}), \mathbb{C}).$$

As in the proof of [3, Theorem 5.1], this equals the map defined in Lemma 3.3.

On the other hand, the second row of Equation 9 is the short exact sequence giving rise to Equation 6. Therefore, the associated map,

$$H^0(X, \Omega_{\mathbb{P}^n}^n(2X)|_X) \rightarrow \mathrm{Hom}(\bigwedge^{n-3} H^0(C, N_{C/X}), \mathbb{C}),$$

defining $\phi_{n-3, [C]}(\bar{\beta})$ is obtained by taking the push-out of the second row of Equation 9 by the last map in the first column. This push-out is canonically isomorphic to the third row of Equation 9. Therefore $\phi_{n-3, [C]}(\bar{\beta})$ comes from the connecting map in cohomology associated to the third row of Equation 9

Finally, the third row of Equation 9 is obtained from the exact sequence in Equation 7 by dualizing and tensoring with ω_C . In particular, using Serre duality, the connecting map for the third row,

$$H^0(C, \omega_C \otimes \bigwedge^{n-2} N_{C/X}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_C) \rightarrow H^1(C, \omega_C \otimes \bigwedge^{n-3} N_{C/X}^\vee),$$

equals the transpose of the map,

$$H^0(C, \bigwedge^{n-3} N_{C/X}) \rightarrow H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N_{X/\mathbb{P}^n}^\vee|_C).$$

Since this is the map used to define $\tilde{\beta}$, the restriction of $\tilde{\beta}$ equals $\phi_{n-3, [C]}(\bar{\beta})$. \square

Proof of Theorem 3.2. Let M be an integral, closed subscheme of $\overline{\mathcal{M}}_{g,0}(X)$ satisfying Hypotheses 3.1. Let $\nu : \widetilde{M} \rightarrow M$ be a desingularization of M . There is an open dense subscheme V of \widetilde{M} on which ν is unramified and over which the pullback family of stable maps is a family of smooth, embedded, $(n-1)$ -normal curves, also assumed nondegenerate if g equals 1. For every point $[C]$ in V , $T_{V, [C]}$ is a subspace of $H^0(C, N_{C/X})$. Therefore $\bigwedge^{n-3} T_{V, [C]}$ is a 1-dimensional subspace of $\bigwedge^{n-3} H^0(C, N_{C/X})$. By Hypothesis 3.1(i), for a general point $[C]$ of V , the image of $\bigwedge^{n-3} T_{V, [C]}$ in $H^0(C, \bigwedge^{n-3} N_{C/X})$ is nonzero.

Associated to the short exact sequence in Equation 7, there is an exact sequence of cohomology,

$$H^0(C, \bigwedge^{n-2} N_{C/\mathbb{P}^n} \otimes N_{X/\mathbb{P}^n}^\vee|_C) \rightarrow H^0(C, \bigwedge^{n-3} N_{C/X}) \xrightarrow{\delta} H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N_{X/\mathbb{P}^n}^\vee|_C). \quad (10)$$

Claim 3.5. $h^0(C, \bigwedge^{n-2} N_{C/\mathbb{P}^n} \otimes N_{X/\mathbb{P}^n}^\vee|_C)$ equals 0.

To prove Claim 3.5, first observe that $\bigwedge^{n-2} N_{C/\mathbb{P}^n}$ is isomorphic to $N_{C/\mathbb{P}^n}^\vee \otimes \bigwedge^{n-1} N_{C/\mathbb{P}^n}$. By adjunction, $\bigwedge^{n-1} N_{C/\mathbb{P}^n}$ is isomorphic to $\omega_C \otimes (\Omega_{\mathbb{P}^n}^n)^\vee|_C \cong \omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)|_C$. Since N_{X/\mathbb{P}^n}^\vee is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-n)|_X$, altogether there is an isomorphism,

$$N_{C/\mathbb{P}^n} \otimes N_{X/\mathbb{P}^n}^\vee|_C \cong \omega_C \otimes N_{C/\mathbb{P}^n}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C.$$

Now N_{C/\mathbb{P}^n}^\vee is a subsheaf of $\Omega_{\mathbb{P}^n}|_C$. Therefore $H^0(C, \omega_C \otimes N_{C/\mathbb{P}^n}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C)$ is a subspace of $H^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C)$. There is an exact sequence,

$$0 \rightarrow \Omega_{\mathbb{P}^n}(1) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0,$$

where the last map is the canonical map. By Hypothesis 3.1(ii), g equals 0 or 1. If g equals 0, $h^0(C, \omega_C)$ equals 0, and thus $h^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C)$ also equals 0. If g equals 1, then ω_C is isomorphic to \mathcal{O}_C . Therefore $H^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C)$ is the kernel of the following map,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(C, \mathcal{O}_{\mathbb{P}^n}(1)|_C).$$

By Hypothesis 3.1(iii), C is nondegenerate, and thus the kernel is trivial. This proves Claim 3.5, both when g equals 0 and when g equals 1.

Because of Claim 3.5, the map δ from Equation 10 is injective. As in the proof of Lemma 3.4, the target of δ is canonically isomorphic to $H^0(C, \Omega_{\mathbb{P}^n}(2X)|_C)^\vee$. Now $\Omega_{\mathbb{P}^n}(2X)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{P}^n}(2n) \cong \mathcal{O}_{\mathbb{P}^n}(n-1)$. By Hypothesis 3.1(ii), C is $(n-1)$ -normal. Thus the following map is injective

$$H^0(C, \Omega_{\mathbb{P}^n}(2X)|_C)^\vee \rightarrow H^0(X, \Omega_{\mathbb{P}^n}(2X)|_X)^\vee.$$

In particular, there exists an element β of $H^0(X, \Omega_{\mathbb{P}^n}(2X)|_X)$ whose associated map $\tilde{\beta}$ is nonzero on $\bigwedge^{n-3} T_{V, [C]}$. By Lemma 3.4, the $(n-3)$ -form $\phi_p(\tilde{\beta})$ is nonzero at $[C]$. Therefore $\phi_p(\tilde{\beta})$ is a nonzero, canonical form on \widetilde{M} , proving \widetilde{M} is not uniruled, cf. [1, Corollary 4.12]. \square

Proof of Theorem 1.5. By a proof similar to that of Theorem 1.3, it suffices to prove there exists no uniruled subvariety M of $\overline{M}_{g,0}(X)$ satisfying Hypothesis 3.1(i)–(iii).

Let M be an integral, uniruled subvariety of $\overline{M}_{g,0}(X)$ satisfying Hypothesis 3.1(i) and (ii), and whose general point parametrizes a smooth, embedded curve. Since M is uniruled, there exists a quasi-projective variety Z and a dominant, generically-finite morphism

$$Z \times \mathbb{P}^1 \rightarrow M.$$

As in the proof of Proposition 2.2, for a general $(n-4)$ -dimensional subvariety Z' of Z , the closed image of $Z' \times \mathbb{P}^1$ is an $(n-3)$ -dimensional, uniruled subvariety M' of M satisfying Hypothesis 3.1(i) and (ii) and containing a general point of M . So M' is uniruled and satisfies Hypothesis 3.1 (i), (ii) and (iv). By Theorem 3.2, M' does not satisfy (iii). Since M' contains a general point of M , also M does not satisfy

(iii). Therefore every integral, closed subvariety M satisfying Hypothesis 3.1(i)–(iii) is not uniruled. \square

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